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THEORETICAL ANALYSIS OF A TYPICAL

SAMPLED-DATA SYSTEM

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**THEORETICAL ANALYSIS OF A TYPICAL  
SAMPLED-DATA SYSTEM**

by

**H. R. McCarley**

**D. A. Project No. 1-B-2-22901-A-204  
AMC Management Structure Code No. 5221.11.146**

**Inertial Systems Branch  
Guidance and Control Laboratory  
Directorate of Research and Development  
U. S. Army Missile Command  
Redstone Arsenal, Alabama**

## ABSTRACT

Closed loop servomechanisms are usually linear feedback systems operating on continuous signals. There is, however, a distinct class of servomechanisms which operates on "sampled data." In these systems the input, or actuating signal, is represented by samples at regular intervals of time with the information carried in the amplitude by the samples.

The analysis of "sampled data systems" presents several features quite different from those found in continuous systems. The error is measured, the correction applied, and then the system passes through a waiting period before the next error measurement. This results in a tendency to over correct for the accumulated error.

This report presents a mathematical stability analysis of a typical sample data servo using the Z transform and root locus. The system was simulated on a Donner 3200 analog computer and the results from this simulation were used to verify the calculated results.

# THEORETICAL ANALYSIS OF A TYPICAL SAMPLED-DATA SYSTEM

## I. INTRODUCTION

A sampled-data control system is a system in which the input is sampled periodically. The samples may then be "operated on" and fed back in the conventional servo method as in Figure 1.

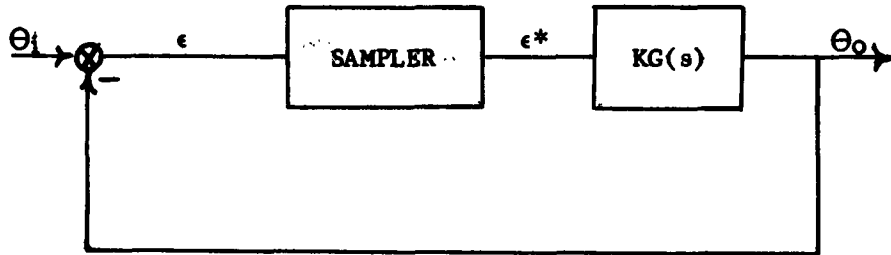


Figure 1

## II. SAMPLER

The sampler converts the continuous signal to a train of regularly spaced pulses with the height representing the value of the input at the sampling time. The sampler is shown in Figure 2 with the input denoted by  $\epsilon$  and output by  $\epsilon^*$ .

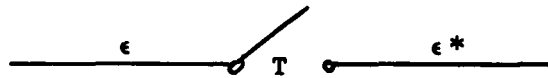


Figure 2

The output of the sampler can be written as

$$\epsilon^*(t) = \epsilon(t) \left[ \sum_{n=0}^{\infty} u_0(t - nT) \right] \quad (1)$$

where the bracketed term is a train of unit impulses occurring every  $T$  seconds ( $T$  is the "sampling period"). When  $t = nT$ ,

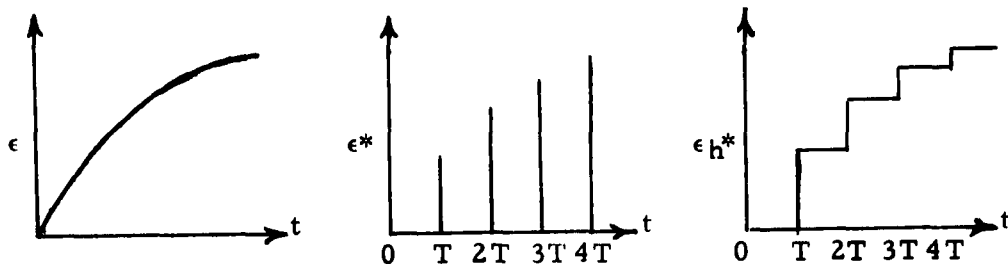


Figure 5

and

$$\epsilon_h^*(s) = \sum_{n=0}^{\infty} \epsilon(nT) e^{-nTs} \left( \frac{1 - e^{-Ts}}{s} \right) \quad (5)$$

or

$$\epsilon_h^*(s) = \epsilon^*(s) \left( \frac{1 - e^{-Ts}}{s} \right) \quad (6)$$

A hold circuit, therefore, may be represented as:

$$H(s) = \frac{\epsilon_h^*(s)}{\epsilon^*(s)} = \frac{1 - e^{-Ts}}{s} \quad (7)$$

where  $T$  is the sampling period.

An electro-mechanical sample-and-hold device is shown in Figure 6.

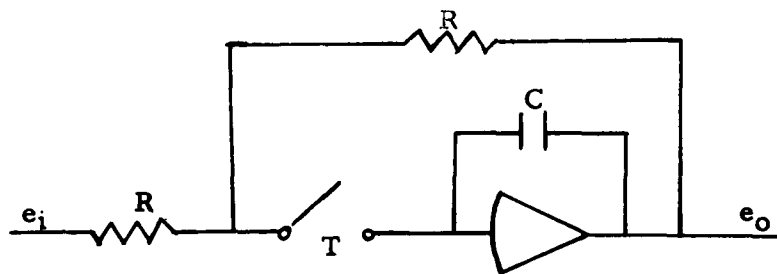


Figure 6

$$\epsilon^*(t) = \sum_{n=0}^{\infty} \epsilon(nT) u_o(t - nT) \quad (2)$$

Taking the Laplace transformation of this equation,

$$\epsilon^*(s) = \sum_{n=0}^{\infty} \epsilon(nT) e^{-nTs} \quad (3)$$

Each sample may be held until the next sample arrives as in Figure 3.

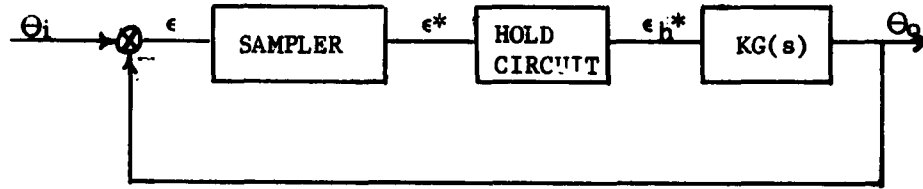


Figure 3

### III. HOLD CIRCUITS

A hold circuit converts a sample train consisting of samples of width  $\Delta t$  into a staircase function as in Figure 4.

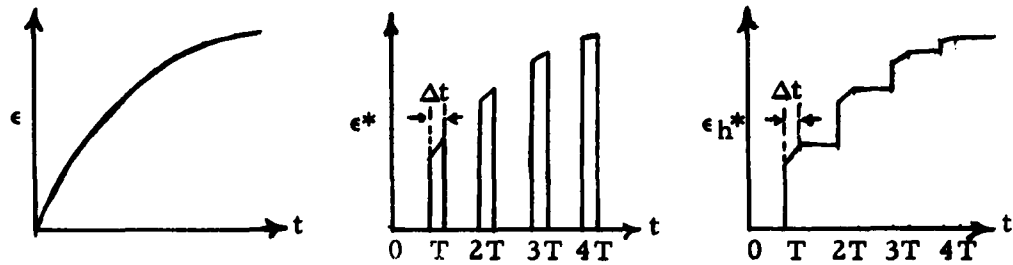


Figure 4

By the principle of superposition,

$$\epsilon_h^*(s) = \sum_{n=0}^{\infty} \epsilon(nT + \Delta t) e^{-nTs} \left( \frac{e^{-\Delta ts} - e^{-Ts}}{s} \right) \quad (4)$$

Assuming  $\Delta t \rightarrow 0$ , the functions in Figure 4 become as those in Figure 5.



or a diode-gate version may be used as shown in Figure 7.

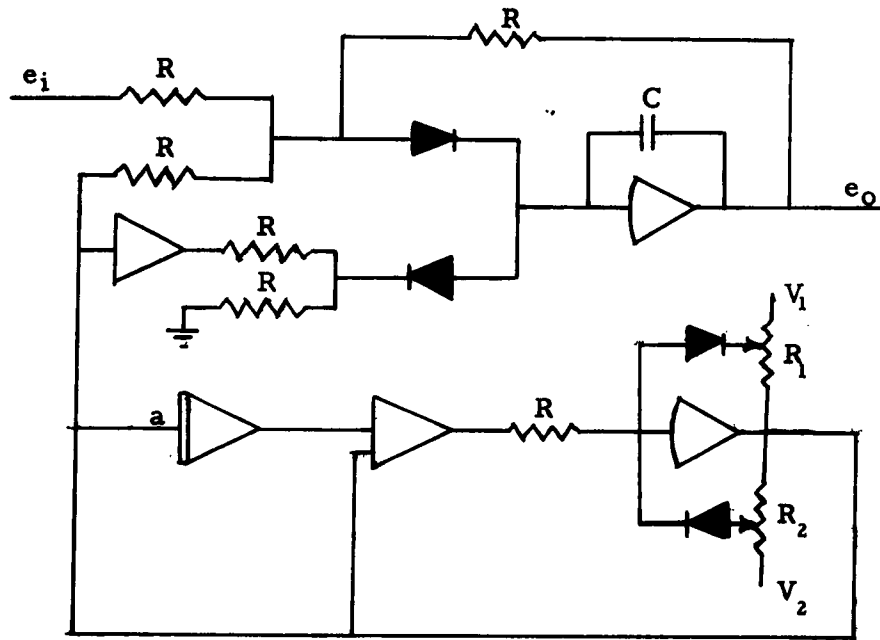


Figure 7

where  $V_1 R_1$ ,  $V_2 R_2$ , and (a) determine the sampling period  $T$ .

The system simulated on the analog computer was a second-order sampled data feedback system with zero-order hold as shown in Figure 8.

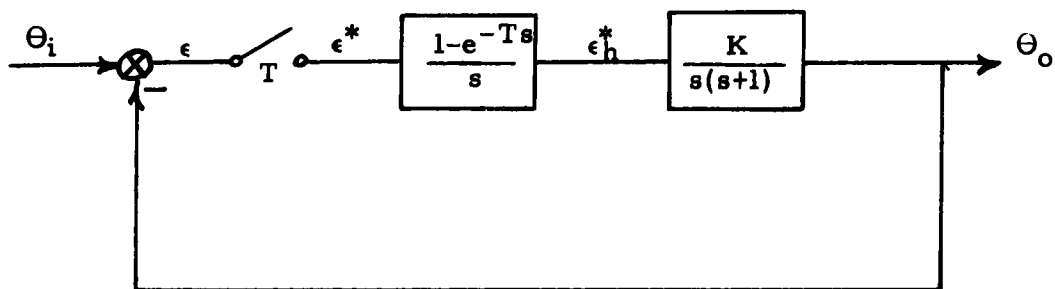


Figure 8

The system was simulated on the computer as in Figure 9.

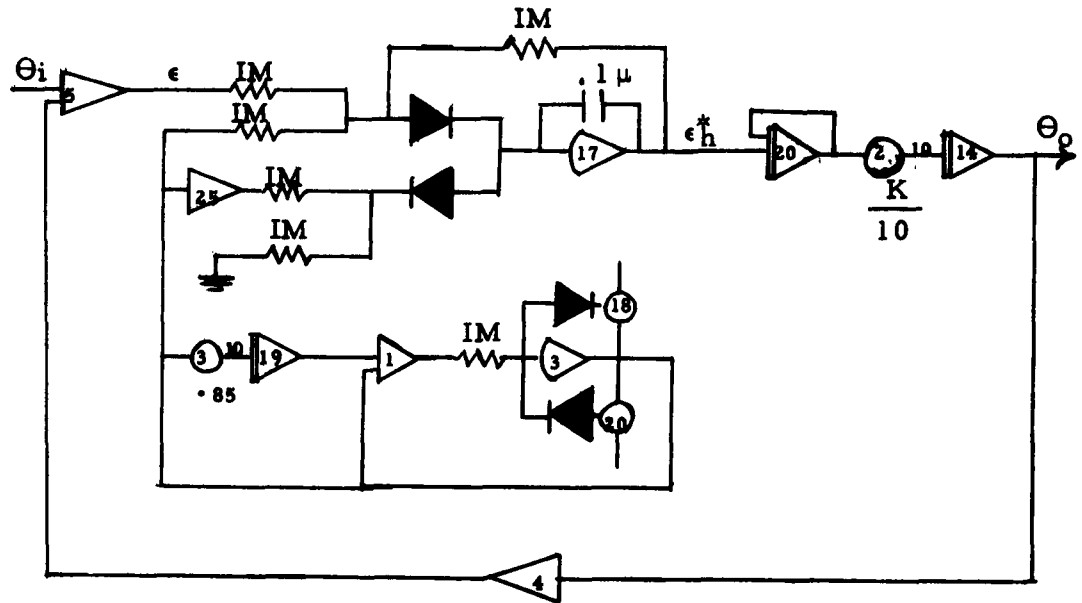


Figure 9

#### IV. Z TRANSFORM

The analysis of a sampled-data system is complicated by the sampler characteristics. The analysis of an over-all system is simplified if the Z-transform is introduced. The Z-transform of a system input describes the behavior of the sampled signal at the sampling instants. The Z-transform of a signal is obtained by making the substitution  $Z = e^{st}$  in the Laplace transformation of the values of the sampled signal at the sampling instants. To contain sufficient information about the signal, the sampling frequency must be at least twice the highest signal frequency. Tables of Z-transforms of common functions are readily available in most servo books.

In a standard S-plane root locus analysis, the characteristic equation is plotted in the S-plane and instability occurs when the roots enter the right-half of the plane. The right-half S-plane maps into the exterior of the unit circle in the Z-plane by the transformation  $Z = e^{st}$ . So the location of the zeros of the characteristic equation  $(1+KG(z))$  in the Z-plane determine stability.

From Figure 8 the open loop transfer function is:

$$KG(s) = \frac{K(1-e^{-sT})}{s^2(s+1)} \quad (8)$$

The corresponding Z-transform is:

$$KG(z) = \frac{KT}{z-1} - \frac{K(1-e^{-T})}{z-e^{-T}} \quad (9)$$

Letting  $T=1$  second and simplifying, the characteristic equation is:

$$1 + KG(z) = 1 + \frac{.368K}{(z-1)} \frac{(z + .72)}{(z-.368)} = 0 \quad (10)$$

Letting  $Z = \alpha + j\beta$ ,

$$\frac{.368K(\alpha + .72 + j\beta)}{(\alpha-1 + j\beta)(\alpha-.368 + j\beta)} = -1 \quad (11)$$

The phase-angle equation is:

$$\tan^{-1} \frac{\beta}{\alpha + .72} - \tan^{-1} \frac{\beta(2\alpha-.368-1)}{\alpha^2-\beta^2-\alpha(1+.368)+.368} = \pi \quad (12)$$

Taking the tangent of equation (12) and simplifying,

$$\frac{1}{\alpha + .72} = \frac{2\alpha-1.368}{\alpha^2-\beta^2-\alpha(1.368)+.368} \quad (13)$$

or,

$$(\alpha + .72)^2 + \beta^2 = (1.368)^2 \quad (14)$$

Therefore, the locus is a circle with its center at  $(-.72, 0)$  and a radius of 1.368. The magnitude equation of equation (11) is:

$$\frac{.368K|z + .72|}{|z-1||z-.368|} = 1 \quad (15)$$

The locus is shown in Figure 10.

The system will become unstable at the values of  $Z$  when equation (14) intersects the unit ( $\alpha^2 + \beta^2 = 1$ ). Solving these equations simultaneously, these values of  $Z$  are:

$$z = .244 \pm j.968 \quad (16)$$

From Equation 15, the maximum allowable gain for stability is  $K = 2.388$ .

## V. RESULTS

Results of the computer simulation may be seen in Figures 11 and 12. The sampling period was set at one second by manipulation of pots 3, 18, and 20, and by manipulation of the gain of integrator by pot 19. A step voltage was applied to amplifier 5. The system became unstable as Pot 2 was varied so that the system gain was 2.4. The root locus analysis of the system showed that a gain greater than 2.388 would cause instability so this study showed excellent agreement between theory and simulation.

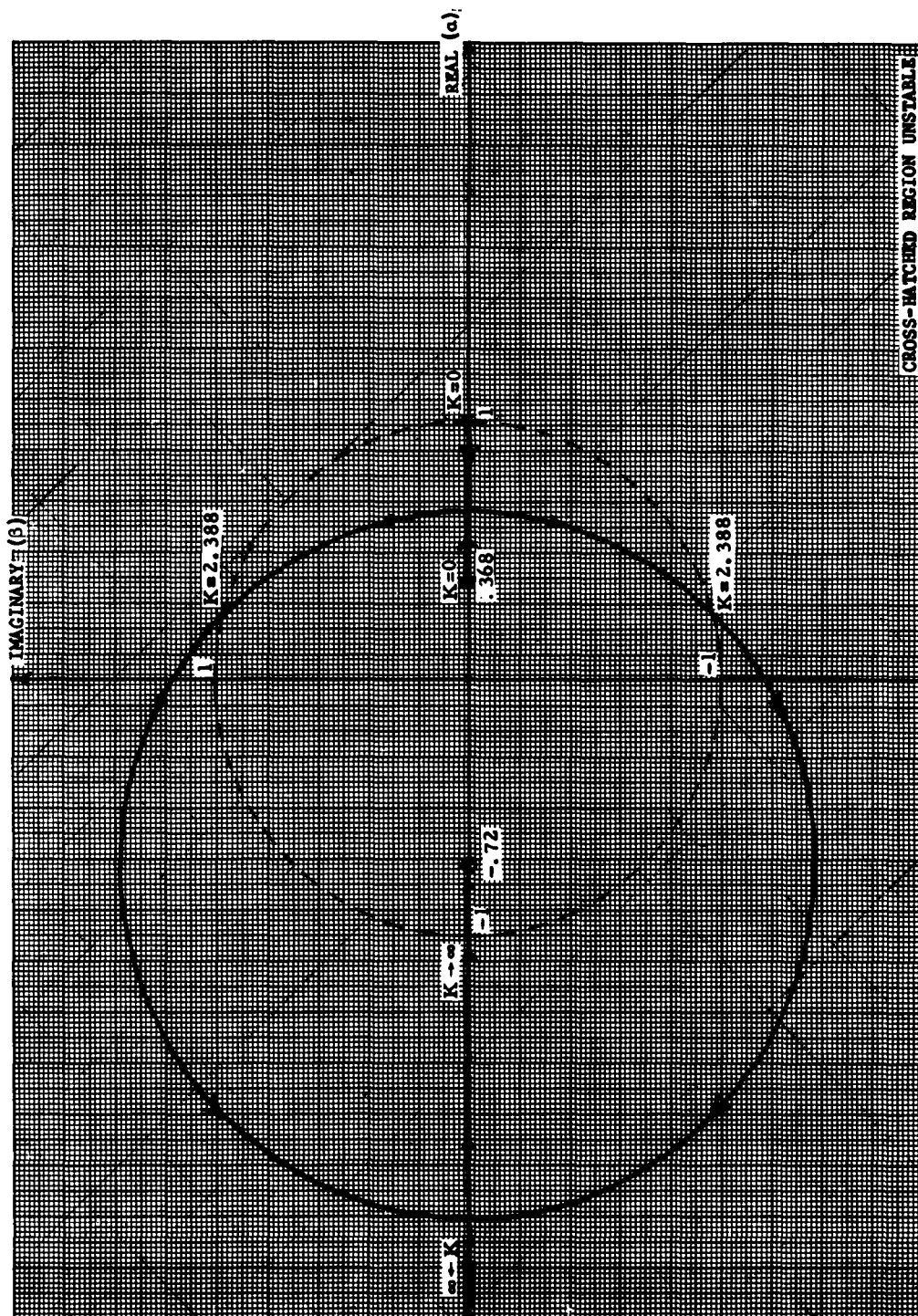


Figure 10. Z-PLANE ROOT LOCUS OF  $\frac{K(1-e^{-s})}{s^2(s+1)}$

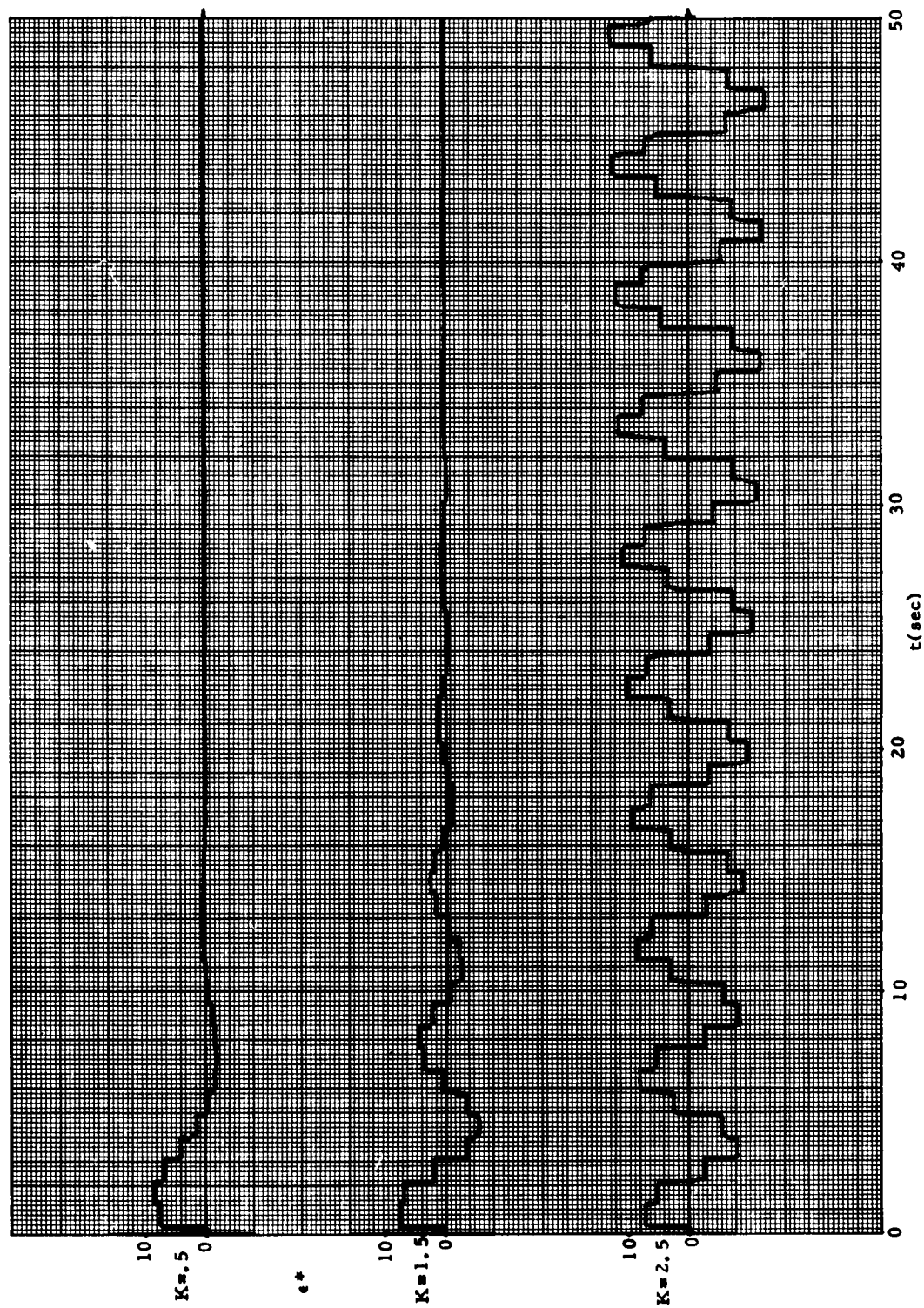


Figure 11. RESPONSE OF SAMPLED-HELD SIGNAL TO STEP INPUT ( $\Theta_1 = 10$  UNITS)

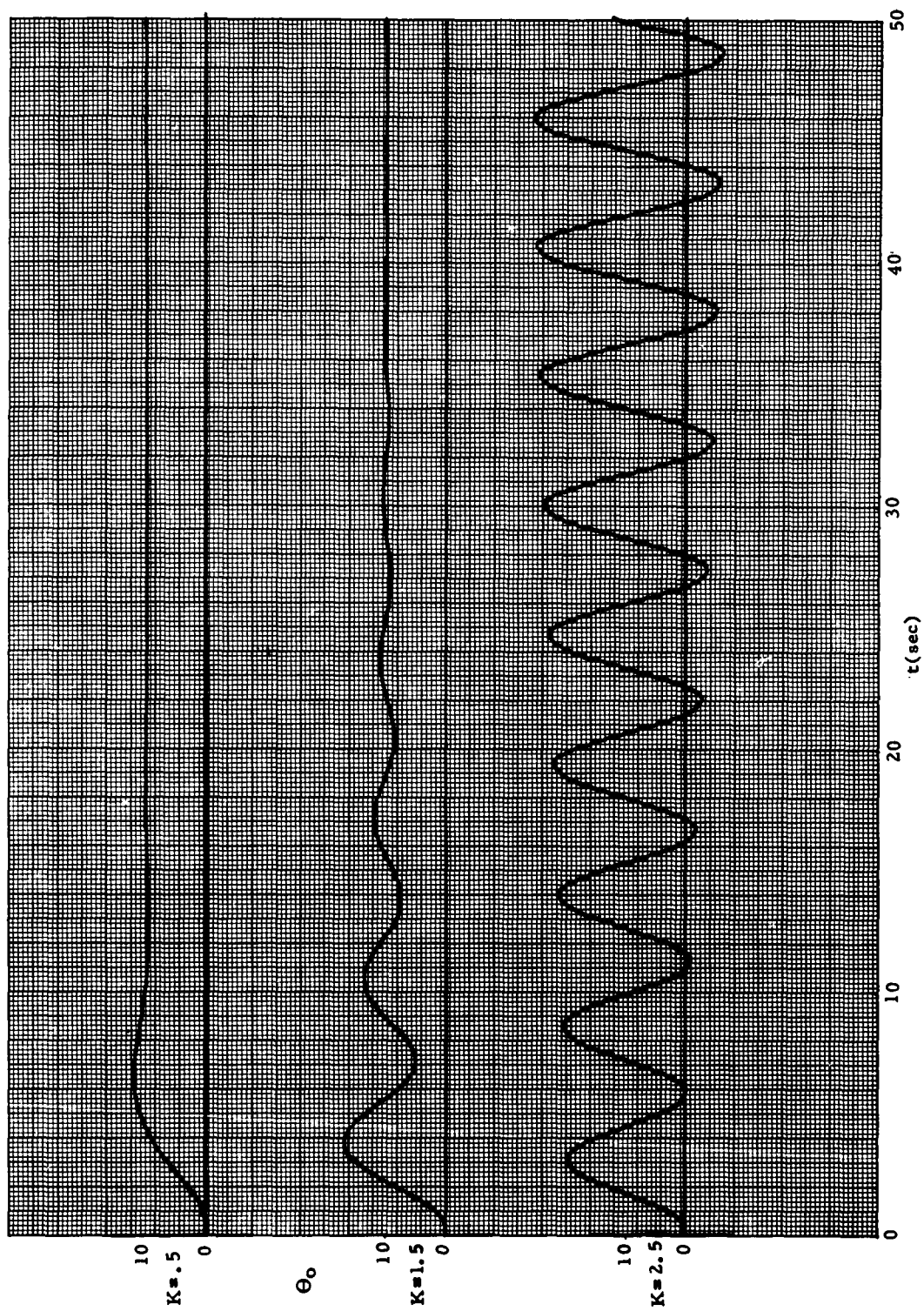
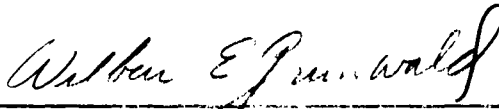


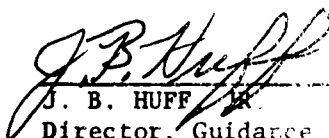
Figure 12. RESPONSE OF OUTPUT TO STEP INPUT ( $\Theta_i = 10$  UNITS)

21 January 1963

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